

# REGULARITY OF THE MINIMIZERS IN THE COMPOSITE MEMBRANE PROBLEM IN $\mathbb{R}^2$

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ABSTRACT. We study the regularity of the minimizers to the problem:

$$\lambda(\alpha, A) = \inf_{u \in H_0^1(\Omega), \|u\|_2=1, |D|=A} \int_{\Omega} |Du|^2 + \alpha \int_D u^2.$$

We prove that in the physical case  $\alpha < \lambda$  in  $\mathbb{R}^2$ , any minimizer  $u$  is locally  $C^{1,1}$  and the boundary of the set  $\{u > c\}$  is analytic where  $c$  is the constant such that  $D = \{u < c\}$  (up to a zero measure set).

## 1. INTRODUCTION

Consider a bounded domain  $\Omega \subset \mathbb{R}^n$  with Lipschitz boundary. Fix  $A$ ,  $0 < A < |\Omega|$  and  $\alpha > 0$ . Our goal is to study the regularity of the minimizers to the problem:

$$(1.1) \quad \lambda(A, \alpha) = \inf_{u \in H_0^1(\Omega), \|u\|_2=1, |D|=A} \int_{\Omega} |Du|^2 + \alpha \int_D u^2.$$

[3] establishes the existence of minimizers and connects (1.1) with a physical problem whose goal is to minimize the first Dirichlet eigenvalue of a body of prescribed shape and mass that has to be constructed out of materials of varying densities. The Euler-Lagrange equation corresponding to (1.1) is

$$(1.2) \quad -\Delta u + \alpha \chi_D u = \lambda(\alpha, A)u.$$

It was proved in [3] that for any optimal configuration  $(u, D)$ , there exists some  $c > 0$  such that  $D = \{u < c\}$  (up to a zero measure set). In fact, the weak uniqueness result in [5] says that this constant  $c$  depends only on  $\Omega, \alpha$  and  $A$ , for almost every  $A$ .

We shall always assume here that  $\alpha < \bar{\alpha}(A)$  where  $\bar{\alpha}$  is a special constant defined in [3]. This condition guarantees that  $\alpha < \lambda(\alpha, A)$ . The physical problem posed in [3] in fact demands that. An elementary consequence of this condition is that  $u$  is strictly superharmonic and hence satisfies the strong minimum principle. So every point in the set  $\{u = c\}$  is a limit point of the set  $\{u < c\}$ , and  $|\{u = c\}| = 0$ .

By a result in [4], for any point  $x_0 \in \{u = c\} \cap \{|Du| > 0\}$ , there exists  $r > 0$  such that the set  $\{u = c\} \cap B_r(x_0)$  is the graph of a real-analytic function. Thus the issue is to

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understand points in the set  $\{u = c\} \cap \{Du = 0\}$ . In [5], these singular points were studied for (1.2) and a blow-up analysis performed to classify the singularities. Such an analysis was done earlier in dimension two in [2] and [9]. The aim of this paper is to study which blow-up solutions of [5] are unstable for the functional (1.1). Ruling out various blow-up solutions leads therefore to improved regularity of the solution  $u$  and also to regularity of the free-boundary  $\{u = c\}$ . In a dumb-bell shaped region  $\Omega$ , it is proved in [3] that one of the lobes fills faster than the other as  $A \rightarrow |\Omega|$ . Thus for certain value of  $A$ , one of the lobes could contain an isolated point of the set  $\{u = c\}$  surrounded solely by points where  $u < c$ . On blow-up we will get a blow-up limit as in [9], in particular the set  $\{u = c\}$  is not regular. Thus in general, even if  $\Omega$  is simply-connected, we do not expect  $\{u = c\}$  to be regular. However, it turns out that  $\partial\{u > c\}$  has better regularity properties. So it may be more natural to view  $\partial\{u > c\}$  as the free-boundary instead of  $\{u = c\}$ . We will therefore denote in this paper

$$(1.3) \quad U = \{u > c\}$$

$$(1.4) \quad \mathcal{F} = \partial U$$

and

$$(1.5) \quad \mathcal{F}^* = \mathcal{F} \cap \{|Du| > 0\}.$$

There is a similarity in spirit between this problem and a problem treated in [8]. The difference being that the problem in our paper has the constraint  $|D| = A$ , which puts complications in the construction of the variations we employ.

It will be easier to study the free functional corresponding to (1.1). We will make both variations in the domain  $D$  and the function  $u$ . We set, for a family of domains  $D(t)$  such that  $|D(t)| = A$ ,

$$(1.6) \quad E(s, t) = \int_{\Omega} |Du + sDv|^2 + \alpha \int_{D(t)} (u + sv)^2 - \lambda \int_{\Omega} (u + sv)^2.$$

Our minimizing assumption then becomes

$$E(s, t) \geq E(0, 0) = 0.$$

In section 2, we find the formula for all first and second derivatives of  $E(s, t)$ . The first derivative of  $E(s, t)$  with regards to  $t$  already played a role in obtaining weak uniqueness in [5]. Pieces of the second variation formula were obtained earlier in [6]. However in order to get any contradiction the full second variation is needed.

We will confine ourselves here to state two consequences of our results. In section 4 we show:

**Theorem 1.1.** *Let  $\Omega \subset \mathbb{R}^2$ ,  $0 < A < |\Omega|$  and  $0 < \alpha < \bar{\alpha}$ . Let  $(u, D)$  be a minimizing configuration. Then  $u \in C^{1,1}(\Omega)$ .*

In contrast, one can construct solutions to the Euler-Lagrange equation (1.2) which fail to have  $C^{1,1}$  bounds [5], and in a related problem, see [1]. We recall that [9] establishes that under (1.2), points  $x_0$  where  $Du(x_0) = 0$  and  $U$  having positive density are isolated.

Such point does exist, see [5]. However, we show that it is not the case for the minimizers of (1.1).

We now turn our attention to the free-boundary  $\mathcal{F} = \partial U$ . We prove in section 6 the following result:

**Theorem 1.2.** *Let  $(u, D)$  be a minimizing configuration. Then the set  $\{u > c\}$  consists of a finite number of connected components whose closures are disjoint. The boundary of each of these components consists of finitely many disjoint, simple and closed real-analytic curves on which  $|Du| > 0$ .*

The proof of theorem 1.2 uses theorem 1.1 and the second variation formula, but no further blow-up arguments are needed. One feature of the proof of theorem 1.2 is the use of global arguments, in particular the use of the Jordan Curve Theorem. Another aspect of this problem is that one first classifies the blow-up limits and then uses the classification to get  $C^{1,1}$  bounds.

It follows from these theorems and a result of [5] that for a minimizing configuration  $(u, D)$ , the 1-dim Hausdorff measure of the set  $\{u = c\}$  is finite.

In the case when  $\Omega$  is simply connected, it follows from [3] that  $D$  is connected. From this fact and the superharmonicity of  $u$ , it is easy to see that each connected component of  $U$  is simply connected and thus has a connected boundary. In this case, the proof of theorem 1.2 simplifies considerably.

Lastly, the situation in higher dimensions is unclear. This is also the case for the problem treated in [8]. In fact, the argument in the proof of step 2, Theorem 8.1 is incomplete because in the notation of [8],

$$\int_{B_1} |Dw_\delta|^2 \approx -\log \delta \rightarrow \infty \text{ as } \delta \rightarrow 0.$$

## 2. SECOND VARIATION FORMULA

We start by defining what we call a regular curve. A curve  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  is regular if it satisfies the following conditions

- i.  $-\infty < a < b < \infty$
- ii. if  $a \leq x < y \leq b$  and  $x \neq a$  or  $y \neq b$ , then  $\gamma(x) \neq \gamma(y)$
- iii.  $\|\gamma\|_{C^2(a,b)}$  is finite
- iv.  $|\gamma'|$  is uniformly bounded away from 0.

If in addition,  $\gamma(a) = \gamma(b)$ , we say it is closed and regular.

If the domain of  $\gamma$  is  $(a, b)$ , we say  $\gamma$  is regular (similarly closed and regular) if the continuous extension of  $\gamma$  to  $[a, b]$  is regular (respectively closed and regular).

We state our key second variation formula in the following lemma.

**Lemma 2.1.** *Let  $J = \cup_{k=1}^n J_k$  be a finite union of open, bounded intervals of  $\mathbb{R}$  and*

$$\gamma = (\gamma_1, \gamma_2) : J \rightarrow \mathcal{F}^*$$

a simple curve which is regular on each interval  $J_k$  and  $\overline{\gamma(J)} \subset \mathcal{F}^*$ . Assume also that  $\text{dist}(\gamma(J_k), \gamma(J_h)) > 0$  for all  $1 \leq h \neq k \leq n$ . For each  $\xi \in J$ , denote by

$$N(\xi) = (N_1(\xi), N_2(\xi))$$

the outward unit normal with respect to  $D$  at  $\gamma(\xi)$ . We also define  $N^*$  to be  $(N_2, -N_1)$  and  $N'$  the first-derivative of  $N$ . Let  $t_0 > 0$  and  $g : J \times (-t_0, t_0) \rightarrow \mathbb{R}$  be a function such that  $g(\cdot, t), g_t(\cdot, t), g_{tt}(\cdot, t) \in C(\overline{J})$  for all  $t \in (-t_0, t_0)$  and

$$(2.1) \quad g(\cdot, 0) \equiv 0$$

$$(2.2) \quad \int_J g(\cdot, t) |\gamma'| + \frac{1}{2} (g(\cdot, t))^2 (N' \cdot N^*) = 0, \forall t \in (-t_0, t_0).$$

Then for any  $v \in H_0^1$  we have

$$(2.3) \quad \left( \int_\Omega |Dv|^2 + \alpha \int_D v^2 - \Lambda \int_\Omega v^2 \right) \int_\gamma (g_t(\gamma^{-1}, 0))^2 |Du| \geq \alpha c \left( \int_\gamma g_t(\gamma^{-1}, 0) v \right)^2.$$

Here  $g_t, g_{tt}$  denote the first and second derivatives of  $g$  with respect to  $t$ .

*Proof.* Reversing the direction of  $\gamma$  if necessary, we will assume without loss of generality that  $\gamma'$  and  $N^*$  have the same direction, i.e

$$\gamma' \cdot N^* = |\gamma'|.$$

For each  $k$ , it is well-known that because  $\gamma$  is  $C^2$  and simple on  $\overline{J_k}$ , there exists a  $\beta_k > 0$  such that the function

$$\phi : J_k \times [-\beta_k, \beta_k] \rightarrow \mathbb{R}^2$$

defined by

$$(x_1, x_2) = \phi(\xi, \beta) = \gamma(\xi) + \beta N(\xi)$$

is injective. Because  $\text{dist}(\gamma(J_h), \gamma(J_k)) > 0$  for  $h \neq k$ , we can find a number  $\beta_0 > 0$  such that  $\phi$  is injective on  $J \times [-\beta_0, \beta_0]$ .

Substituting  $t_0$  by a smaller positive number if necessary, we can assume that

$$\|g\|_{L^\infty(J)} < \beta_0.$$

Let

$$K = D \setminus \phi(J \times (-\beta_0, 0]).$$

Define for each  $t \in (-t_0, t_0)$

$$(2.4) \quad D(t) = K \cup \{\phi(\xi, \beta) \mid \xi \in J, \beta < g(\xi, t)\}$$

We can compute  $A(t)$ , the measure of  $D(t)$  by the formula

$$(2.5) \quad A(t) = |D| + \int_J \int_0^{g(\xi, t)} J(\xi, \beta, t) d\beta d\xi$$

where

$$\begin{aligned} J(\xi, \beta) &= \begin{vmatrix} \gamma'_1 + \beta N'_1 & N_1 \\ \gamma'_2 + \beta N'_2 & N_2 \end{vmatrix} \\ &= |(\gamma' \cdot N^*) + \beta(N' \cdot N^*)| \\ &= ||\gamma'| + \beta(N' \cdot N^*)|. \end{aligned}$$

Because  $\|\gamma\|_{C^2(J)} < \infty$ , we have  $\|N' \cdot N^*\|_{L^\infty(J)} < \infty$ . Again by considering a smaller positive number  $t_0$ , we can assume that

$$\|g\|_{L^\infty(J)} \|N' \cdot N^*\|_{L^\infty(J)} \leq \theta.$$

Thus,

$$|\gamma'| \geq \theta \geq |\beta| |N' \cdot N^*|$$

for all  $\xi \in J$  and  $|\beta| \leq \|g\|_{L^\infty(J)}$  and so,

$$J = |\gamma'| + \beta(N' \cdot N^*).$$

Substituting into the formula for  $A(t)$  in (2.5) we have

$$\begin{aligned} A(t) &= A + \int_J \int_0^{g(\xi, t)} |\gamma'| + \beta(N' \cdot N^*) \\ &= A + \int_J g(., t) |\gamma'| + \frac{1}{2} (g(., t))^2 (N' \cdot N^*) \\ &= A \quad (\text{due to (2.2)}). \end{aligned}$$

We also have for later reference,

$$\begin{aligned} A'(t) &= \int_J g_t |\gamma'| + g g_t (N' \cdot N^*) \\ A''(t) &= \int_J g_{tt} |\gamma'| + (g g_{tt} + g_t^2) (N' \cdot N^*). \end{aligned}$$

More generally, if  $F$  is a continuous function from  $\mathbb{R}^2$  to  $\mathbb{R}$ , then

$$\int_{D(t)} F - \int_D F = \int_J \int_0^{g(\xi, t)} F(\phi(\xi, \beta)) J(\xi, \beta) d\beta d\xi$$

and so from the Fundamental Theorem of Calculus,

$$(2.6) \quad \frac{\partial}{\partial t} \int_{D(t)} F = \int_J g_t(., t) F(\phi(., g(., t))) J(., g(., t)).$$

Define the functional

$$E(s, t) = \int_{\Omega} (Du + sDv)^2 + \alpha \int_{D(t)} (u + sv)^2 - \lambda \int_{\Omega} (u + sv)^2.$$

We will compute all second-derivatives of  $E$  with respect to  $s$  and  $t$ .

First, the second derivative of  $E$  with respect to  $s$ ,

$$(2.7) \quad \frac{\partial^2 E}{\partial^2 s}(s, t) = 2 \left( \int_{\Omega} |Dv|^2 + \alpha \int_{D(t)} v^2 - \lambda \int_{\Omega} v^2 \right)$$

Applying (2.6) with  $F = (u + sv)^2$ , we have the first derivative of  $E$  with respect to  $t$ ,

$$(2.8) \quad \frac{\partial E}{\partial t}(s, t) = \alpha \int_J g_t(., t) (u(\phi(., g(., t))) + sv(\phi(., g(., t))))^2 J(., g(., t)) d\xi$$

To compute the second derivative of  $E$  with respect to  $t$ , differentiating (2.8) and noting that

$$\frac{\partial}{\partial t} (u(\phi(., g(., t))))^2 = 2u(\phi(., g(., t))) Du(\phi(., g(., t))) \cdot Ng_t$$

we have

$$\begin{aligned} \frac{\partial^2 E}{\partial^2 t}(0, t) &= \alpha \frac{\partial}{\partial t} \int_J u(\phi(., g))^2 g_t(|\gamma'| + g(N' \cdot N^*)) d\xi \\ &= \alpha \int_J u(\phi(., g))^2 (g_{tt} |\gamma'| + (gg_{tt} + g_t^2)(N' \cdot N^*)) \\ &\quad + \alpha \int_J 2u(\phi(., g)) Du(\phi(., g)) \cdot Ng_t^2(|\gamma'| + g(N' \cdot N^*)) \end{aligned}$$

When  $t = 0$ ,  $Du(\phi(., g(., 0))) = Du(\gamma(.)) = |Du(\gamma(.))| N(.)$  and so,

$$\begin{aligned} \frac{\partial^2 E}{\partial^2 t}(0, 0) &= \alpha c^2 A''(0) + 2\alpha c \int_J g_t(., 0)^2 |Du(\gamma(.))| |\gamma'| \\ (2.9) \quad &= 2\alpha c \int_{\gamma} (g_t(\gamma^{-1}, 0))^2 |Du|. \end{aligned}$$

To compute the mixed second derivative of  $E$ , differentiating (2.8) with respect to  $s$  we have

$$\begin{aligned} \frac{\partial^2 E}{\partial s \partial t}(0, t) &= 2\alpha \int_J g_t u(\phi(., g)) v(\phi(., g)) J(., g(., t)) \\ \frac{\partial^2 E}{\partial s \partial t}(0, 0) &= 2\alpha c \int_J g_t(., 0) v(\gamma(.)) |\gamma'| \\ (2.10) \quad &= 2\alpha c \int_{\gamma} g_t(\gamma^{-1}, 0) v. \end{aligned}$$

For any value of  $s$  and  $t \in (-t_0, t_0)$ ,  $u + sv \in H_0^1$  and  $|D(t)| = A$ , so from the definition of  $(u, D)$  we have that  $E(0, 0)$  is a minimum value of  $E(s, t)$ . Consequently,

$$\frac{\partial^2 E}{\partial^2 s}(0, 0) \frac{\partial^2 E}{\partial^2 t}(0, 0) \geq \left( \frac{\partial^2 E}{\partial s \partial t}(0, 0) \right)^2.$$

Substituting formula (2.9), (2.7) and (2.10) into this inequality we obtain the desired result.  $\square$

Notice that in the formula (2.3), only value of  $g_t$  is present. Hence we would like to know for what kind of function  $g_t$  we can find  $g$  that satisfies all hypotheses of the last lemma.

**Lemma 2.2.** *Let  $J$  and  $\gamma$  be the same as in the Lemma 2.1. Assume that  $h : \gamma \rightarrow \mathbb{R}$  is a bounded, continuous function that satisfies*

$$\int_{\gamma} h = 0.$$

Then for all  $v \in H_0^1(\Omega)$  and  $a \in \mathbb{R}$  we have

$$(2.11) \quad \left( \int_{\Omega} |Dv|^2 + \alpha \int_D v^2 - \lambda \int_{\Omega} v^2 \right) \int_{\gamma} h^2 |Du| \geq \alpha c \left( \int_{\gamma} h(v - a) \right)^2.$$

*Proof.* Define  $N$  as in the Lemma 2.1. Also define  $g : J \times (-t_0, t_0) \rightarrow \mathbb{R}$  by

$$g(., t) = \frac{2t(h \circ \gamma) |\gamma'|}{|\gamma'| + \sqrt{|\gamma'|^2 + 2t(h \circ \gamma) |\gamma'| (N' \cdot N^*)}}.$$

Since  $|\gamma'|$  is bounded below by  $\theta > 0$  and  $h, (N' \cdot N^*)$  are bounded above, we can choose  $t_0$  small enough so that  $g$  is well-defined in  $J \times (-t_0, t_0)$ . Clearly  $g(., 0) \equiv 0$  and  $g, g_t, g_{tt}$  are continuous functions in  $J$ . It also satisfies the equation

$$(2.12) \quad g |\gamma'| + \frac{1}{2} g^2 (N' \cdot N^*) = t(h \circ \gamma) |\gamma'|$$

and so for all  $t \in (-t_0, t_0)$ ,

$$\int_J g |\gamma'| + \frac{1}{2} g^2 (N' \cdot N^*) = t \int_J (h \circ \gamma) |\gamma'| = t \int_{\gamma} h = 0.$$

Differentiating (2.12) with respect to  $t$  and letting  $t = 0$  we obtain

$$g_t(., 0) = h \circ \gamma.$$

Since  $g$  satisfies all the required hypothesis of the Lemma 2.1, we can apply it and obtain

$$\left( \int_{\Omega} |Dv|^2 + \alpha \int_D v^2 - \lambda \int_{\Omega} v^2 \right) \int_{\gamma} h^2 |Du| \geq \alpha c \left( \int_{\gamma} hv \right)^2.$$

Due to the fact that

$$\int_{\gamma} h = 0,$$

we have

$$\int_{\gamma} hv = \int_{\gamma} h(v - a).$$

The conclusion then follows.  $\square$

3. A REGULARITY CRITERION FOR  $\partial\{u > c\}$ 

**Lemma 3.1.** *Let  $P$  be a point on  $\mathcal{F} = \partial\{u > c\}$ . Suppose that for each  $k \in \mathbb{Z}^+$ , there exist a positive number  $r_k$ , a bounded and open interval  $J_k$  and a regular curve  $\gamma_k : J_k \rightarrow \mathcal{F}^*$  that satisfy the following conditions*

$$\begin{aligned} r_1 &> r_2 > \cdots \rightarrow 0 \\ \overline{\gamma_k(J_k)} &\subset \mathcal{F}^* \cap B_{r_k}(P) \setminus \overline{B_{r_{k+1}}(P)} \end{aligned}$$

Then we must have

$$\sum_{k=1}^{\infty} \int_{\gamma(J_k)} \frac{1}{|Du|} < \infty.$$

*Proof.* Assume without loss of generality that  $P$  is the origin. Assume also that  $J_k \cap J_h = \emptyset$  for all  $k \neq h$  so we can use one notation  $\gamma$  for all  $\gamma_k$ . We will use the following notation

$$J_{k,m} = \begin{cases} J_k \cup J_{k+1} \cup \cdots \cup J_m, & \text{if } m \geq k \\ \emptyset, & \text{otherwise.} \end{cases}$$

Assume that

$$\sum_{k=1}^{\infty} \int_{\gamma(J_k)} \frac{1}{|Du|} = \infty.$$

We will derive a contradiction.

Let  $V$  be a smooth, radial function in  $\mathbb{R}^2$  such that  $V$  is decreasing in  $|x|$  and

$$(3.1) \quad \begin{cases} V(x) = 2, & |x| = 0 \\ 2 > V(x) > 1, & |x| \in (0, 1/2) \\ 1 > V(x) > 0, & |x| \in (1/2, 1) \\ V(x) = 0, & |x| \geq 1 \end{cases}$$

For each  $k \in \mathbb{Z}^+$ , define  $v_k(x) = V(x/r_k)$ . It is easy to verify that when  $r_k$  is small enough,

$$\int_{\Omega} |Dv_k|^2 = \int_{\Omega} |DV|^2$$

and so for any  $k$  large enough

$$(3.2) \quad \int_{\Omega} |Dv_k|^2 + \alpha \int_D |v_k|^2 - \lambda \int_{\Omega} |v_k|^2 < \int_{\Omega} |DV|^2 < \infty.$$

We will drop the subscript  $k$  from the rest of the proof. We list here values of  $v - 1$  for easy reference later,

$$(3.3) \quad \begin{cases} v(x) - 1 = 1, & |x| = 0 \\ 1 > v(x) - 1 > 0, & |x| \in (0, r_k/2) \\ 0 > v(x) - 1 > -1, & |x| \in (r_k/2, r_k) \\ v(x) - 1 = -1, & |x| \geq r_k. \end{cases}$$



Because  $J_k$  and  $|\gamma'|$  are bounded,  $\gamma(J_k)$  is of finite length. We also have  $|Du|$  is uniformly bounded away from 0 on  $\gamma(J)$  since  $\overline{\gamma(J)} \subset \mathcal{F}^*$ . Together with the fact that  $\gamma(J_{0,k-1}) \subset {}^c B_{r_k}$ , we have

$$-\infty < \int_{\gamma(J_{0,k-1})} \frac{v-1}{|Du|} = - \int_{\gamma(J_{0,k-1})} \frac{1}{|Du|} < 0.$$

Choose an  $m$  such that  $r_m < r_k/2$ . From the facts that  $v(x) - 1 > 0$  in  $B_{r_m}$ ,  $\gamma(J_l) \subset B_{r_m}$  for all  $l \geq m$  and  $v(x) - 1 \rightarrow 1$  as  $|x| \rightarrow 0$  we have

$$\int_{\gamma(J_{m,\infty})} \frac{v-1}{|Du|} \sim \int_{\gamma(J_{m,\infty})} \frac{1}{|Du|} = \infty.$$

Consequently, there must be a number  $l \geq m$  such that

$$\int_{\gamma(J_{m,l-1})} \frac{v-1}{|Du|} \leq - \int_{\gamma(J_{0,k-1})} \frac{v-1}{|Du|} < \int_{\gamma(J_{m,l})} \frac{v-1}{|Du|}.$$

Choose a subinterval  $J'_l \subset J_l$  such that

$$\int_{\gamma(J_{m,l-1})} \frac{v-1}{|Du|} + \int_{\gamma(J'_l)} \frac{v-1}{|Du|} = - \int_{\gamma(J_{0,k-1})} \frac{v-1}{|Du|}.$$

In other words, we have

$$\int_{\gamma(J^k)} \frac{v-1}{|Du|} = 0.$$

where  $J^k = J_{0,k-1} \cup J_{m,l-1} \cup J'_l$ .

We can now apply the Lemma 2.2 to  $J^k$ ,  $\gamma$ ,  $v$ ,  $a = 1$  and  $h = (v-1)/|Du|$ , and obtain

$$\begin{aligned} \int_{\Omega} |DV|^2 \int_{\gamma(J^k)} \frac{(v-1)^2}{|Du|} &\geq \alpha c \left( \int_{\gamma(J^k)} \frac{(v-1)^2}{|Du|} \right)^2 \\ \int_{\Omega} |DV|^2 &\geq \alpha c \int_{\gamma(J^k)} \frac{(v-1)^2}{|Du|} \\ &\geq \alpha c \int_{\gamma(J_{0,k-1})} \frac{(v-1)^2}{|Du|} \\ &\geq \alpha c \int_{\gamma(J_{0,k-1})} \frac{1}{|Du|} \quad (v-1 = -1 \text{ on } \gamma(J_{0,k-1}) \subset {}^c B_{r_k}). \end{aligned}$$

Let  $k$  go to  $\infty$  we have

$$\int_{\Omega} |DV|^2 \geq \alpha c \int_{\gamma(J_{0,\infty})} \frac{1}{|Du|} = \infty$$

which is a contradiction.

So we must have

$$\sum_{k=1}^{\infty} \int_{\gamma(J_k)} \frac{1}{|Du|} < \infty$$

as desired.  $\square$

Next, we prove a direct consequence of the last lemma. Informally, it says that if the set  $\partial\{u > c\} \cap \{|Du| > 0\}$  is big enough around a point of  $\partial\{u > c\}$ , then at this point,  $|Du| > 0$ .

**Lemma 3.2.** *Let  $P$  be a point on  $\mathcal{F} = \partial\{u > c\}$ . Suppose that there are numbers  $K \in \mathbb{Z}$  and  $\sigma > 0$  such that for each  $k \geq K$ , there exists a regular curve  $\gamma_k : J_k \rightarrow \mathcal{F}^*$  with the following properties*

$$\begin{aligned} \overline{\gamma_k(J_k)} &\subset \mathcal{F}^* \cap B_{2^{-k}}(P) \setminus \overline{B_{2^{-(k+1)}}(P)} \\ \mathcal{H}^1(\gamma_k(J_k)) &= \int_{J_k} |\gamma'_k| > \sigma 2^{-k}. \end{aligned}$$

Then  $|Du(P)| > 0$ .

*Proof.* Assume that  $Du(P) = 0$ . To derive a contradiction, it is enough to show that

$$\sum_{k=K}^{\infty} \int_{\gamma(J_k)} \frac{1}{|Du|} = \infty$$

and use the last lemma.

From a result in [7] and the fact that  $\Delta u \in L^\infty$ , there exists some positive constant  $C$  such that for all  $x \in \Omega$ ,

$$|Du(x)| = |Du(x) - Du(P)| \leq C |x - P| \log(1/|x - P|).$$

Thus,

$$\begin{aligned} \sum_{k=K}^{\infty} \int_{\gamma(J_k)} \frac{1}{|Du|} &\geq \frac{1}{C} \sum_{k=K}^{\infty} \int_{\gamma(J_k)} \frac{1}{|x - P| \log(1/|x - P|)} \\ &\geq \frac{1}{C} \sum_{k=K}^{\infty} \frac{\sigma 2^{-k}}{2^{-k} \log(2^k)} \\ &= \frac{1}{C} \frac{\sigma}{\log 2} \sum_{k=K}^{\infty} \frac{1}{k} \\ &= \infty. \end{aligned}$$

$\square$

#### 4. $C^{1,1}$ REGULARITY

We now apply the regularity criterion from the last section to show that if the set  $\{u > c\}$  has positive density at a point of the set  $\partial\{u > c\}$ , then at that point  $|Du| > 0$ .

**Theorem 4.1.** *Let  $P$  be a point on  $\partial\{u > c\}$ . Assume that there exist  $\beta, r_0 > 0$  such that*

$$|\{u > c\} \cap B_r(P)| \geq \beta r^2$$

*for all  $0 < r < r_0$ . Then  $|Du(P)| > 0$ .*

*Proof.* Without loss of generality, let  $P$  be the origin. Assume that  $Du(P) = 0$ . For each  $r > 0$ , define

$$v_r(x) = \frac{c - u(rx)}{r^2}.$$

Also define  $I(r)$  to be the supremum of lengths of all regular curves with closures in the set

$$\{v_r = 0\} \cap \{|Dv_r| > 0\} \cap B_1 \setminus \overline{B_{1/2}}.$$

We show that there exist some  $r_0 > 0$  and  $\sigma > 0$  such that  $I(r) > \sigma$  for all  $0 < r < r_0$ .

Assume that it is not the case, then there exists a sequence  $r_k \rightarrow 0$  such that  $I(r_k) \rightarrow 0$ . As a consequence of the Theorem 3.1 in [5], two possibilities arise.

- (1) A subsequence of  $v_{r_k}/T(r_k)$  converges to a non-zero, homogeneous of degree 2 harmonic function where

$$T(r) = \frac{1}{r^2} \left( \frac{1}{2\pi r} \int_{\partial B_r} (c - u)^2 \right)^{1/2}.$$

- (2) A subsequence of  $v_{r_k}$  converges to a homogeneous solution of degree 2 of the equation

$$\Delta v = c(\lambda - \alpha)\mathcal{X}_{\{v \geq 0\}} + c\lambda\mathcal{X}_{\{v < 0\}}.$$

We consider case (1) first. Without loss of generality, we can assume that  $v_{r_k}/T(r_k)$  converges to  $v(x) = x_1 x_2$  in  $C^{1,\delta}$  as  $k \rightarrow \infty$ . We will hereafter denote  $v_{r_k}$  by  $v_k$  and  $T(r_k)$  by  $T_k$ .

Let  $\epsilon$  be any number in  $(0, 1/8)$ . It can be verified easily that

$$Q_1 = [1/2 + \epsilon, 1 - \epsilon] \times [-\epsilon, \epsilon]$$

is a subset of the set  $B_1 \setminus \overline{B_{1/2}}$ . We have for any  $x_1 \in [1/2 + \epsilon, 1 - \epsilon]$ ,

- i. The first-derivative with respect to  $x_2$ ,  $v_2(x_1, \cdot) = x_1 \in (1/2, 1)$ .
- ii.  $v(x_1, -\epsilon) = -\epsilon x_1 \leq -\epsilon/2$  and  $v(x_1, \epsilon) = \epsilon x_1 \geq \epsilon/2$ .

Since  $v_k/T_k \rightarrow v$  in  $C^{1,\delta}$ , we can choose some  $N$  such that for all  $k > N$ ,

$$|v_k/T_k - v|_\infty < \epsilon/4, |(v_k)_2/T_k - v_2| < 1/4 \text{ in } Q_1.$$

It follows that for all  $k > N$ ,

- (1)  $5/4 > (v_k)_2/T_k > 1/4$  on  $[1/2 + \epsilon, 1 - \epsilon] \times [-\epsilon, \epsilon]$ .
- (2)  $v_k(x_1, -\epsilon)/T_k \leq -\epsilon/4$  and  $v_k(x_1, \epsilon)/T_k \geq \epsilon/4$ .

Consequently, for each  $x_1$ , there is exactly one value of  $x_2$  such that  $v_k(x_1, x_2) = 0$ . Denote this value by  $\tau_k(x_1)$  and define  $\gamma_k(x_1) = (x_1, \tau_k(x_1))$ . Since  $5/4 > (v_k)_2/T_k > 1/4$ , doing implicit differentiation we have  $-\infty < \tau'_k < \infty$  and so  $1 \leq |\gamma'_k| < \infty$ .  $\gamma_k$  is also clearly the

boundary of a connected component of the set  $\{v_k < 0\}$  since a neighborhood below it is an open subset of the set  $\{v_k < 0\}$ . The length of  $\gamma_k$  is at least

$$(1 - \epsilon) - (1/2 + \epsilon) = 1/2 - 2\epsilon > 1/4.$$

This implies that  $I(r_k) > 1/4$  for all  $k > N$ , contradicting our assumption that  $I(r_k) \rightarrow 0$ .

In the second case, we can also assume that  $v_k$  converges in  $C^{1,\delta}$  to a homogeneous solution of degree 2 of the equation

$$\Delta v = c(\lambda - \alpha)\mathcal{X}_{\{v \geq 0\}} + c\lambda\mathcal{X}_{\{v < 0\}}.$$

Since

$$|\{u > c\} \cap B_r| \geq \beta r^2,$$

in terms of  $v_k$  we have

$$|\{v_k < 0\} \cap B_1| \geq \beta.$$

Letting  $k$  go to  $\infty$  we obtain

$$|\{v \leq 0\} \cap B_1| \geq \beta.$$

From the Lemma 1.2 in [9], we know that either the set  $\{v = 0\} \cap \{Dv = 0\} = \{0\}$  or  $v$  is of the form  $c(\lambda - \alpha)x_1^2/2$  after a rotation. Because

$$|\{c(\lambda - \alpha)x_1^2/2 \leq 0\} \cap B_1| = 0,$$

contradicting the positive density condition for  $v$  above, we must have then

$$\{v = 0\} \cap \{Dv = 0\} = \{0\}.$$

Since

$$|\{v \leq 0\} \cap B_1| \geq \beta,$$

$v$  is superharmonic and  $v$  is homogeneous, there exists a point  $z$  such that  $|z| = 1$  and  $v(tz) = 0$  for all  $t \in [0, 1]$ . Assume that  $z = (1, 0)$ . We also have  $Dv(1/2, 0) \neq 0$  due to the fact that

$$\{v = 0\} \cap \{Dv = 0\} = \{0\}.$$

Because  $|v_2(1/2, 0)| = |Dv(1/2, 0)| \neq 0$ , we can assume without loss of generality that  $u_2(1/2, 0) > 0$ . Now, arguing similarly to the first case, we obtain  $I(r_n) > \sigma$  for some  $\sigma > 0$  when  $n$  large enough, contradicting our assumption that  $I(r_n) \rightarrow 0$ .

Thus, in all cases, there exist  $\sigma > 0$  and  $r_0 > 0$  such that  $I(r) > \sigma$  for all  $0 < r < r_0$ . In other words, for each  $r < r_0$ , there exists a regular curve of length at least  $\sigma r$  with closure in the set

$$\{v_r = 0\} \cap \{Dv_r \neq 0\} \cap B_1 \setminus \overline{B_{1/2}}.$$

In terms of  $u$ , it means for all  $0 < r < r_0$ , there exists a regular curve of length at least  $\sigma r$  with closure in the set

$$\mathcal{F}^* \cap B_r \setminus \overline{B_{r/2}}.$$

Applying the Lemma 3.2 we have  $|Du(P)| > 0$ , contradicting the assumption that  $Du(P) = 0$ . So  $|Du(P)| > 0$ .  $\square$

**Corollary 4.2.**  $u \in C^{1,1}(\Omega)$ .

*Proof.* It is clear that  $u$  is  $C^{1,1}$  at points in  $\{u \neq c\}$  or  $\{u = c\} \cap \{|Du| > 0\}$ . Assume that there exists a point  $P \in \{u = c\} \cap \{|Du| = 0\}$  at which  $u$  is not  $C^{1,1}$ . In other words,

$$\limsup_{r \rightarrow 0} \sup_{|x-P| < r} \frac{|u(rx+P) - c|}{r^2} \rightarrow \infty.$$

From the Lemma 3.18 in [5], there must exist  $\beta, r_0 > 0$  such that

$$|\{u > c\} \cap B_r(P)| \geq \beta r^2 \text{ for all } 0 < r < r_0.$$

However, the Theorem 4.1 then implies that  $|Du(P)| > 0$ , a contradiction. Thus  $u \in C^{1,1}(\Omega)$ .  $\square$

## 5. REGULARITY OF CONNECTED COMPONENTS OF $\{u > c\}$

We first prove the following lemma.

**Lemma 5.1.** *Let  $L \subset \mathbb{R}^2$  be a connected set. Furthermore, assume that for any  $P \in L$ , there exists  $r > 0$  such that the set  $B_r(P) \cap L$  is a regular curve. Then given any pair  $S, Q \in L$ , there exists a regular curve in  $L$  with  $S, Q$  as two end points.*

*Proof.* Define  $L_S$  to be the set of points  $R \in L$  such that there exists a regular curve in  $L$  with  $S, R$  as two endpoints. We will show that  $L_S$  is non-empty, closed and open. Because  $L$  is connected, it means  $L_S = L$  and the conclusion follows.

Let  $r > 0$  be a number such that  $L \cap B_r(S)$  is a regular curve. Obviously, any point in this set is a point in the set  $L_S$  as well. So  $L_S$  is non-empty.

Assume that  $R \in L_S$ . Let  $r > 0$  be a number such that  $B_r(R) \cap L$  is a regular curve. Because there is a regular curve connecting  $S$  and  $R$ , it is easy to see that for any  $R' \in B_r(R) \cap L$ , we can truncate or extend that regular curve to obtain a new regular curve connecting  $S$  and  $R'$ . Thus,  $B_r(R) \cap L \subset L_S$ . Since it is true for all  $R \in L_S$ ,  $L_S$  must be open.

Arguing similarly we have, if  $R \in {}^c L_S$ , then there exists  $r > 0$  such that  $B_r(R) \cap L \subset {}^c L_S$ . In other words,  $L_S$  is closed.  $\square$

Next, we prove our first result about the structure of the set  $\partial\{u > c\} \cap \{|Du| > 0\}$ .

**Theorem 5.2.** *If  $\mathcal{F}_1$  is a connected component of  $\mathcal{F} = \partial U$ , then either  $|Du| > 0$  at every point of  $\mathcal{F}_1$ , or  $|Du| \equiv 0$  on  $\mathcal{F}_1$ .*

*Proof.* Assume that  $\mathcal{F}_1$  contains at least one point where  $|Du| > 0$ . Let  $L$  be a connected component of the set  $\mathcal{F}_1 \cap \{|Du| > 0\}$ .  $L$  must be non-empty by definition.

Since for each  $S \in L$ , there exists a number  $r > 0$  such that  $B_r(S) \cap \partial\{u > c\}$  is a simple, analytic curve where  $|Du| > 0$ ,  $L$  has to be open.

We will show that  $L$  is closed as well. Choose any convergent sequence  $\{P_n\}$  in  $L$ . Because  $\mathcal{F}_1$  is a connected component of  $\mathcal{F}$ ,  $\mathcal{F}_1$  is closed. Thus, there exists some  $P \in \mathcal{F}_1$  such that

$$P_n \rightarrow P \in \mathcal{F}_1 \text{ as } n \rightarrow \infty.$$

Pick any  $r_0 < |P_1 - P|$  (here  $P_1$  is the first point in the sequence  $\{P_n\}$ ). For any  $0 < r < r_0$ , there exists some  $P_n$  such that  $|P_n - P| < r/2$ . From Lemma 5.1 we have there exists a regular curve  $\gamma : [0, l] \rightarrow L$  such that  $\gamma(0) = P_1$  and  $\gamma(l) = P_n$ . Define

$$\begin{aligned} a &= \inf \{s \in [0, l] \mid \gamma([s, l]) \subset B_r(P)\} \\ b &= \inf \{s \in [a, l] \mid |\gamma(s) - P| = r/2\}. \end{aligned}$$

The existence of  $a < b \in (0, l)$  is justified because  $\gamma$  is a regular curve and  $|\gamma(0) - P| > r$  while  $|\gamma(l) - P| < r/2$ . It can also be verified easily that

$$\begin{aligned} |\gamma(a) - P| &= r, |\gamma(b) - P| = r/2 \\ \gamma((a, b)) &\subset B_r(P) \setminus \overline{B_{r/2}(P)} \\ \mathcal{H}^1(\gamma((a, b))) &\geq r/2. \end{aligned}$$

Pick some  $\epsilon > 0$  small so that the length of the segment  $\gamma((a + \epsilon, b - \epsilon))$  is at least  $r/3$ . It also follows from the above argument that

$$\overline{\gamma((a + \epsilon, b - \epsilon))} \subset \mathcal{F}^* \cap B_r(P) \setminus \overline{B_{r/2}(P)}.$$

Since we can do it for all  $r < r_0$ , the Lemma 3.2 then implies that  $|Du(P)| > 0$ . Consequently, there exists  $r_1 > 0$  such that  $B_{r_1}(P) \cap \mathcal{F}$  is a regular curve where  $|Du| > 0$ . Since  $\mathcal{F}_1$  is a connected component of  $\mathcal{F}$  and  $P \in \mathcal{F}_1$ , the whole curve  $B_{r_1}(P) \cap \mathcal{F}$  must be in  $\mathcal{F}_1$ . Pick some  $P_n$  such that  $|P_n - P| < r_1$ . It is clear that  $P_n$  has to be in the curve  $B_{r_1}(P) \cap \mathcal{F}$ . But because  $P_n \in L$ ,  $|Du| > 0$  on  $B_{r_1}(P) \cap \mathcal{F}$  and  $L$  is connected, the whole curve  $B_{r_1}(P) \cap \mathcal{F}$  has to be in  $L$ . In particular,  $P \in L$ . Since  $\{P_n\}$  is an arbitrary convergent sequence in  $L$ , it implies that  $L$  is closed.

We have proved that  $L$  is non-empty, open and closed. Because  $\mathcal{F}_1$  is connected, we have  $L = \mathcal{F}_1$ . In other words  $|Du| > 0$  for every point on  $\mathcal{F}_1$ .  $\square$

**Lemma 5.3.** *Let  $U_1$  be a connected component of  $U$  and  $\mathcal{F}_1$  a connected component of  $\partial U_1$  such that  $|Du| > 0$  on  $\mathcal{F}_1$ . Then  $\mathcal{F}_1$  is also a connected component of  $\mathcal{F}$ .*

*Proof.* Let  $P$  be any point on  $\mathcal{F}_1$ . Because  $|Du(P)| > 0$ , there exists  $r > 0$  such that the set  $B_r(P) \cap \mathcal{F}$  is a regular curve that divides  $B_r(P)$  into two disjoint connected regions, one where  $u < c$  and one where  $u > c$ . It is easy to see that the connected region where  $u > c$  is a subset of  $U_1$  and so  $B_r(P) \cap \mathcal{F} \subset \mathcal{F}_1$ . Now for each point in  $\mathcal{F}_1$ , pick a ball like before and consider the union  $V$  of all these balls. Clearly  $V$  is open and  $V \cap \mathcal{F} = \mathcal{F}_1$ . Hence,  $\mathcal{F}_1$  is a connected component of  $\mathcal{F}$ .  $\square$

Next we show that the set  $\{|Du| > 0\}$  is dense in the boundary of each connected component of  $U$ , improving Lemma 2.3 in [5].

**Lemma 5.4.** *If  $U_1$  is a connected component of  $U$ , then  $\partial U_1 = \overline{\partial U_1 \cap \{|Du| > 0\}}$ .*

*Proof.* Let  $P$  be a point on  $\partial U_1$  such that  $Du(P) = 0$ . We will show that for any  $\epsilon > 0$ , there exists a point  $Q \in \partial U_1$  such that  $|P - Q| < \epsilon$  and  $|Du(Q)| > 0$ .

Since  $P \in \partial U_1$ , we can choose a point  $S \in U_1$  such that  $|P - S| < \epsilon/2$ . Define

$$r = \sup \{s \mid B_s(S) \subset U_1\}.$$

It is obvious that  $0 < r \leq |P - S| < \epsilon/2$  and  $\partial B_r(S) \cap \partial U_1 \neq \emptyset$ . Let  $Q$  be any point of the set  $\partial B_r(S) \cap \partial U_1$ . Because  $u$  is superharmonic and  $Q$  is a boundary minimum point of  $u$  in the set  $\overline{B_r(z)}$ , from the Hopf's Lemma we have  $|Du(Q)| > 0$ . We also have easily  $|P - Q| < \epsilon$  due to the facts that  $|P - S| < \epsilon/2$  and  $|S - Q| = r < \epsilon/2$ .  $\square$

**Lemma 5.5.** *Let  $U_1$  be a connected component of  $U$  and  $P$  a point on  $\partial U_1$  such that  $|Du(P)| = 0$ , then for any  $r > 0$ , there exists a connected component  $\mathcal{F}_1$  of  $\partial U_1$  such that  $|Du| > 0$  in  $\mathcal{F}_1$  and  $\mathcal{F}_1 \subset B_r(P)$ .*

*Proof.* Let's assume that  $P$  is the origin. First, we show that there exists an  $r' > 0$  such that for any connected component  $\mathcal{F}_1$  of  $\mathcal{F}$  where  $|Du| > 0$ , if  $\mathcal{F}_1 \cap {}^c B_r \neq \emptyset$ , then  $\mathcal{F}_1 \subset {}^c B_{r'}$ . In other words, if  $\mathcal{F}_1$  contains a point outside  $B_r$ , then the whole component  $\mathcal{F}_1$  has to stay outside  $B_{r'}$ .

If it is not the case, then for any  $r' > 0$ , there exists some connected component  $\mathcal{F}_1$  of  $\mathcal{F}$  such that  $|Du| > 0$  on  $\mathcal{F}_1$ ,  $\mathcal{F}_1 \cap {}^c B_r \neq \emptyset$  and  $\mathcal{F}_1 \cap B_{r'} \neq \emptyset$ . It means for any  $k > \log_2(1/r)$ , there exists a connected component  $\mathcal{F}_1$  of  $\mathcal{F}$  such that  $\mathcal{F}_1 \cap {}^c B_{2^{-k}} \neq \emptyset$ ,  $\mathcal{F}_1 \cap B_{2^{-k-1}} \neq \emptyset$  and  $|Du| > 0$  on  $\mathcal{F}_1$ . Choose  $P_1, P_2 \in \mathcal{F}_1$  such that  $|P_1| \geq 2^{-k}$ ,  $|P_2| < 2^{-(k+1)}$ . From the Lemma 5.1, there exists a regular curve connecting  $P_1$  and  $P_2$ . Arguing as in the Lemma 5.2, we can find a smaller regular piece of this curve of length at least  $2^{-k}/3$  in the set

$$\mathcal{F}_1 \cap B_{2^{-k}} \setminus \overline{B_{2^{-k-1}}}.$$

Since we can do it for all  $k > \log_2(1/r)$ , applying the Lemma 3.2 we can conclude that  $|Du(P)| > 0$ , contradicting our hypothesis on  $P$ . The existence of  $r'$  then follows.

Now using the Lemma 5.4, we can choose a point  $Q \in \partial U_1$  such that  $Q \in B_{r'}$  and  $|Du(Q)| > 0$ . Let  $\mathcal{F}_1$  be the connected component of  $\partial U_1$  that contains  $Q$ . It follows from what we just proved above that  $\mathcal{F}_1 \subset B_r$ . To show that  $|Du| > 0$  on  $\mathcal{F}_1$ , just note that because  $\mathcal{F}_1$  is a connected component of  $\partial U_1$  and  $\partial U_1 \subset \mathcal{F}$ , there exists a connected component  $\mathcal{F}'_1$  of  $\mathcal{F}$  such that  $\mathcal{F}_1 \subset \mathcal{F}'_1$ . Because  $|Du(Q)| > 0$  and  $Q \in \mathcal{F}_1 \subset \mathcal{F}'_1$ , applying the Lemma 5.2 we have  $|Du| > 0$  on  $\mathcal{F}'_1$ .  $\square$

Next, we prove a lemma about the geometric structure of regular connected components of  $\mathcal{F}$ .

**Lemma 5.6.** *If  $\mathcal{F}_1$  is a connected component of  $\mathcal{F}$  such that  $|Du| > 0$  on  $\mathcal{F}_1$ , then  $\mathcal{F}_1$  is a closed and regular curve.*

*Proof.* Pick any point  $P$  on  $\mathcal{F}_1$ . Consider the ODE

$$(5.1) \quad \gamma'(t) = \frac{(Du(\gamma(t)))^*}{|Du(\gamma(t))|}, \gamma(0) = P$$

where  $\gamma$  is a function from  $[0, \infty)$  to  $\mathcal{F}_1$ . Here, as in the section 2,  $N^*$  denotes the vector obtained from rotating  $N$  clockwise an angle of  $\pi/2$ .

First, it is easy to see that if a solution  $\gamma$  exists up to some time  $t_0$ , then we can extend that solution to  $t_0 + \epsilon$  for some  $\epsilon > 0$ . Indeed, because  $\mathcal{F}_1$  is closed, so  $\gamma(t_0) \subset \mathcal{F}_1$ . Since  $\mathcal{F}_1$  is regular, there exists some  $r > 0$  such that  $\mathcal{F}_1 \cap B_r(\gamma(t_0))$  is the graph of a analytic function. Thus, we can extend  $\gamma$  to some time  $t_0 + \epsilon$ . Consequently, this solution  $\gamma$  exists for all time.

Define

$$T = \sup \{t \mid \gamma((0, t)) \text{ is simple}\}.$$

Because  $B_r(P) \cap \mathcal{F}_1$  is a simple curve for some  $r > 0$  small,  $T \geq r > 0$ . We also have since  $|\gamma'| = 1$  that the length of  $\gamma((0, T))$  is exactly  $T$ . We will show  $T < \infty$  by proving that  $\mathcal{H}^1(\mathcal{F}_1) < \infty$ .

Since  $\mathcal{F}_1 \subset \mathcal{F}^*$ , for each point  $Q \in \mathcal{F}_1$ , there exists  $r > 0$  such that  $B_r(Q) \cap \mathcal{F}_1$  is an analytic curve. It implies that  $\mathcal{H}^1(B_r(Q) \cap \mathcal{F}_1) < \infty$ . Because  $\mathcal{F}_1$  is closed and bounded, we can cover  $\mathcal{F}_1$  by a finite number of such balls and so  $\mathcal{H}^1(\mathcal{F}_1) < \infty$ .

We will show that there exists a time  $T' \in [0, T)$  such that  $\gamma(T') = \gamma(T)$ .

Choose a decreasing sequence of  $\{t_k\}$  that converges to  $T$ . Define

$$a_k = \inf \{a \in [0, t_k) \mid \gamma(a) = \gamma(t) \text{ for some } t \in (a, t_k)\}$$

and

$$b_k = \inf \{b \in (a_k, t_k) \mid \gamma(a_k) = \gamma(b)\}.$$

The existence of  $a_k$  is justified from the fact that  $\gamma([0, t_k))$  is not simple. The existence of  $b_k \geq a_k$  follows the continuity of  $\gamma$ . We show that actually  $b_k > a_k$ . Indeed, since there exists an  $r > 0$  such that  $B_r(\gamma(a_k)) \cap \mathcal{F}_1$  is a simple curve, there is no  $t \in (a_k, a_k + r)$  such that  $\gamma(a_k) = \gamma(t)$  and so  $b_k \geq a_k + r > a_k$ .

We also have other properties of  $a_k, b_k$

- i.  $\{a_k\}$  is increasing.
- ii.  $a_k \leq T \leq b_k < t_k$ .

Passing to a subsequence if necessary, assume that  $a_k \rightarrow T'$  as  $k \rightarrow \infty$ . It is trivial that  $b_k \rightarrow T$  and  $\gamma(T') = \gamma(T)$ . All we need to do now is to show that  $T' < T$ . Indeed since there exists  $r > 0$  such that  $B_r(\gamma(T)) \cap \mathcal{F}_1$  is a simple curve,  $\gamma((T - r, T + r))$  is a simple curve. When  $k$  is large enough,  $b_k \in [T, T + r)$  and consequently  $a_k \leq T - r$ . Thus  $T' \leq T - r < T$ . We also note that there exists no other pair  $(a, b) \neq (T', T)$  with  $0 \leq a < b \leq T$  such that  $\gamma(a) = \gamma(b)$ .

If  $T' \neq 0$ , then as a consequence of the result above, for all  $r > 0$  small, the set  $\mathcal{F}_1 \cap B_r(\gamma(T'))$  consists of three disjoint arcs  $\gamma(T' - r, T']$ ,  $\gamma[T', T' + r)$  and  $\gamma(T - r, T]$  that intersect at an endpoint  $\gamma(T')$ , contradicting the fact that  $B_r(\gamma(T')) \cap \mathcal{F}_1$  is a regular curve when  $r > 0$  is small. Thus,  $T' = 0$ .



To show that  $\mathcal{F}_1 = \gamma([0, T])$ , we argue the same way as in the Lemma 5.3 to show that there exists an open set  $V$  such that  $V \cap \gamma([0, T]) = \gamma([0, T])$  and note that  $\mathcal{F}_1$  is connected.  $\square$

If  $\mathcal{F}_1$  is a connected component of  $\mathcal{F}$  such that  $|Du| > 0$ , then by the Lemma 5.6 above, we know that  $\mathcal{F}_1$  is a closed and regular curve. Using the Jordan Curve Theorem (see for example [10]), we know that  $\mathcal{F}_1$  divides  $\mathbb{R}^2$  into two separate regions, an inside region and an outside region. We will denote the inside region as  $I(\mathcal{F}_1)$  and the outside region  $O(\mathcal{F}_1)$ .

**Theorem 5.7.** *If  $U_1$  is a connected component of  $U$ , then  $|Du| > 0$  on  $\partial U_1$ .*

*Proof.* Without loss of generality, let's assume that  $0 \in \partial U_1$  and  $Du(0) = 0$ . Choose  $r > 0$  such that

$$r < \frac{c}{\|Du\|_\infty} \text{ and } U_1 \not\subset B_r$$

From the Lemma 5.5 we have that there exists some connected component  $\mathcal{F}_1$  of  $\partial U_1$  such that  $|Du| > 0$  in  $\mathcal{F}_1$  and  $\mathcal{F}_1 \subset B_r$ . By the Lemmas 5.3 and 5.6,  $\mathcal{F}_1$  is closed and regular. Thus, following the remark preceding this theorem, we can talk about the inside region  $I(\mathcal{F}_1)$  and outside region  $O(\mathcal{F}_1)$ . We know that both  $I(\mathcal{F}_1)$  and  $O(\mathcal{F}_1)$  are open and connected. Furthermore,  $I(\mathcal{F}_1)$  is bounded while  $O(\mathcal{F}_1)$  is unbounded. Because  $\mathcal{F}_1 \subset B_r$ , we can connect any point in  ${}^c B_r$  to a point far away by a line that does not intersect  $\mathcal{F}_1$  and so  ${}^c B_r \subset O(\mathcal{F}_1)$ . Consequently,  $I(\mathcal{F}_1) \subset B_r$ .

Because  $U_1$  is connected, we must have either  $U_1 \subset I(\mathcal{F}_1)$  or  $U_1 \subset O(\mathcal{F}_1)$ . Since  $I(\mathcal{F}_1) \subset B_r$  and  $U_1 \not\subset B_r$ , we cannot have  $U_1 \subset I(\mathcal{F}_1)$ . Thus  $U_1 \subset O(\mathcal{F}_1)$ . Let  $P$  be a point on  $\mathcal{F}_1$ . There exists  $r' > 0$  such that  $B_{r'}(P) \cap \mathcal{F}_1$  is a regular curve that divides  $B_{r'}(P)$  into two disjoint connected regions, one where  $u > c$  and another where  $u < c$ . Because  $P$  is a boundary point of  $U_1$ , it is clear that the region where  $u > c$  must be a subset of  $U_1$  and so, a subset of  $O(\mathcal{F}_1)$ . It implies that the region where  $u < c$  is a subset of  $I(\mathcal{F}_1)$ . Thus  $u < c$  for some point in  $I(\mathcal{F}_1)$ . However, since  $u$  is superharmonic,  $u$  cannot have an interior minimum in the set  $I(\mathcal{F}_1)$ . Thus, there must be a point  $Q \in I(\mathcal{F}_1)$  such that  $u(Q) = 0$ . In other words,  $Q \in \partial\Omega$ . But then from the facts that  $Q \in I(\mathcal{F}_1) \subset B_r$  and

$$r < \frac{c}{\|Du\|_\infty}$$

we must have

$$|u(Q) - u(0)| < r \|Du\|_\infty < c,$$

contradicting the fact that  $u(Q) = 0$  and  $u(0) = c$ .

In other words,  $|Du| > 0$  at every point on  $\partial U_1$ .  $\square$

## 6. REGULARITY OF $\partial \{u > c\}$

At the end of last section, we have proved that  $|Du| > 0$  on the boundary of each component of  $U$ . It might still happen that connected components of  $U$  accumulate to a point where  $|Du| = 0$ . For example, connected components of  $U$  consists a sequence

of smaller and smaller balls that converge to a point. In this section, we prove that this scenario cannot happen. Indeed,  $U$  only has a finite number of connected components.

**Lemma 6.1.** *Let  $U_1$  be a connected component of  $U$ . Then there exists a unique connected component  $\mathcal{F}_1$  of  $\partial U_1$  such that  $U_1 \subset I(\mathcal{F}_1)$ . We will say that  $\mathcal{F}_1$  surrounds  $U_1$ .*

*Proof.* Pick any point  $P \in U_1$ . Define

$$d = \sup \{|P - x| \mid x \in U_1\}.$$

Clearly, there exists a point  $Q \in \partial U_1$  such that  $|P - Q| = d$ . Assume without loss of generality that  $P$  is the origin and  $Q = (d, 0)$ . It is easy to see that  $U_1$  has to be on the left-side of the line  $x_1 = d$  due to the definition of  $d$ . From this and the fact that  $(d, 0) \in \partial U_1$ , we have the outward unit normal with respect to  $U_1$  at  $Q$  has to be  $e_1$ . Let  $\mathcal{F}_1$  be the connected component of  $\partial U_1$  that contains  $Q$ . Note that  $e_1$  will also be the outward unit normal to  $I(\mathcal{F}_1)$  and so, there must exist some  $\epsilon > 0$  such that  $(d, d - \epsilon) \times \{0\} \subset I(\mathcal{F}_1)$  and  $(d + \epsilon, d) \times \{0\} \subset O(\mathcal{F}_1)$ .

Let  $r > 0$  such that  $B_r(Q) \cap \mathcal{F}_1$  is a regular curve that divides  $B_r(Q)$  into two regions,  $u > c$  and  $u < c$ . Since  $U_1$  is connected and  $Q \in \partial U_1$ , the region  $u > c$  is a subset of  $U_1$ . Because the outward unit normal vector at  $Q$  to this curve is  $e_1$ , by choosing a smaller  $\epsilon$  if necessary, we have  $u < c$  on one of two sets  $(d, d - \epsilon) \times \{0\}$ ,  $(d, d + \epsilon) \times \{0\}$  and  $u > c$  on the other. Because the set where  $u > c$  must be a subset of  $U_1$ , it has to be on the left-side of  $(d, 0)$  and thus, it has to be  $(d, d - \epsilon) \times \{0\}$ . Hence  $U_1 \cap I(\mathcal{F}_1) \neq \emptyset$ . But  $U_1$  is connected, so  $U_1 \subset I(\mathcal{F}_1)$ .

Assume there is another connected component  $\mathcal{F}_2$  of  $\partial U_1$  such that  $U_1 \subset I(\mathcal{F}_2)$ . It is easy to derive that  $\mathcal{F}_1 \subset \overline{I(\mathcal{F}_2)}$  and  $\mathcal{F}_2 \subset \overline{I(\mathcal{F}_1)}$ . Consequently,  $\mathcal{F}_2 \equiv \mathcal{F}_1$ .  $\square$

**Lemma 6.2.** *Let  $U_1$  be a connected component of  $U$  and  $\mathcal{F}_1$  the connected component of  $\partial U_1$  that surrounds  $U_1$ . Assume further that  $u \geq c/2$  in the convex hull of  $I(\mathcal{F}_1)$ . Then*

$$\int_{\mathcal{F}_1} \frac{1}{|Du|} \geq \frac{1}{C_1}$$

where  $C_1 = \|u\|_{C^{1,1}(\{u \geq c/2\})}$ .

*Proof.* Without loss of generality, assume that  $u$  attains its maximum value in  $U_1$  at the origin. Let  $P$  be the point on  $\mathcal{F}_1$  such that

$$|P| = \max \{|x| \mid x \in \mathcal{F}_1\}.$$

Let  $x$  be any point on  $\mathcal{F}_1$ . Since both  $x$  and 0 belongs to the convex hull of  $I(\mathcal{F}_1)$ ,  $u \geq c/2$  on the line segment that connects 0 and  $x$ . Thus, we have

$$\begin{aligned} |Du(x)| &= |Du(x) - Du(0)| \\ &\leq C_1 |x| \\ &\leq C_1 |P|. \end{aligned}$$

Because  $P \in \mathcal{F}_1$  and  $0 \in I(\mathcal{F}_1)$ , the line connecting  $P$  and  $0$  has to intersect with  $\mathcal{F}_1$  at another point  $Q$  and  $0$  is between  $P$  and  $Q$ . Clearly, the length of  $\mathcal{F}_1$  is greater than the length of the line segment  $PQ$  which is greater than  $|P|$ . Thus,

$$\int_{\mathcal{F}_1} \frac{1}{|Du|} > \frac{|P|}{C_1|P|} = \frac{1}{C_1}.$$

□

**Lemma 6.3.** *Let  $P$  be a point in  $\mathcal{F}$  such that  $Du(P) = 0$ . Then for any  $r > 0$ , there exists a connected component  $U_1$  of  $U$  such that  $U_1 \subset B_r(P)$ .*

*Proof.* First, we show that there exists a number  $r' > 0$  such that if  $U_1$  is any connected component of  $U$  with  $U_1 \cap {}^c B_r \neq \emptyset$ , then  $U_1 \subset B_{r'}$ . Indeed if it is not the case, then for any  $k \in \mathbb{Z}$  such that  $2^k < r$ , there exists a connected component  $U_1$  of  $U$  such that

$$U_1 \cap {}^c B_{2^k}(P) \neq \emptyset \text{ and } U_1 \cap B_{2^{k-1}}(P) \neq \emptyset.$$

Let  $\mathcal{F}_1$  be a connected component of  $\partial U_1$  such that  $\mathcal{F}_1$  surrounds  $U_1$ . We must have then that

$$\mathcal{F}_1 \cap {}^c B_{2^k}(P) \neq \emptyset \text{ and } \mathcal{F}_1 \cap B_{2^{k-1}}(P) \neq \emptyset.$$

Arguing as in the Lemma 5.2, we can derive the existence of a regular curve in

$$(B_{2^k}(P) \setminus \overline{B_{2^{k-1}}(P)}) \cap \mathcal{F}_1$$

and of length at least  $2^k/3$ . Since we can do it for all  $k$  such that  $2^k < r$ , from the Lemma 3.2 we have  $|Du(P)| > 0$ , contradicting our hypothesis on  $P$ . The existence of  $r'$  follows then.

Because  $P \in \mathcal{F}$ , there must exist a connected component  $U_1$  of  $U$  such that

$$U_1 \cap B_{r'}(P) \neq \emptyset.$$

The result above then guarantees that  $U_1 \subset B_r(P)$ . □

**Theorem 6.4.**  $|Du| > 0$  on  $\partial\{u > c\}$ .

*Proof.* Assume that  $\mathcal{F}$  contains some point where  $Du = 0$ . Without loss of generality, let's assume that that point is the origin. Clearly this point is not on the boundary of any connected component of  $U$ , as a consequence of our result in section 5.

Pick  $r_1 > 0$  such that  $u > c/2$  in the set  $B_{r_1}$ . From the previous lemma, there exists a connected component  $U_1$  of  $U$  such that  $U_1 \subset B_{r_1}$ . Since  $0 \notin \partial U_1$ , there exists  $r_2 > 2$  such that  $B_{r_2} \subset {}^c U_1$ . Choose a connected component  $U_2$  of  $U$  such that  $U_2 \subset B_{r_2}$ . Repeating for each  $k$  we find a number  $r_k > 0$  and a connected component  $U_k$  of  $U$ . Let  $\mathcal{F}_k$  be the connected component of  $\partial U_k$  that surrounds  $U_k$ . Clearly,  $\mathcal{F}_k$  is a regular curve and the

convex hull of  $I(\mathcal{F}_k)$  is inside  $B_{r_1}$ . We have from definitions and the Lemma 6.2 that

$$(6.1) \quad r_1 > r_2 > \cdots \rightarrow 0$$

$$(6.2) \quad \overline{\mathcal{F}_k} \subset \mathcal{F}^* \cap B_{r_k} \setminus \overline{B_{r_{k+1}}}$$

$$(6.3) \quad \sum_{k=1}^{\infty} \int_{\mathcal{F}_k} \frac{1}{|Du|} > \sum_{k=1}^{\infty} \frac{1}{C_1} = \infty.$$

Applying the Lemma 3.1 we reach a contradiction.

Thus,  $|Du| > 0$  on  $\partial U$ . □

We combine all our results into the following statement.

**Theorem 6.5.** *Let  $\Omega \subset \mathbb{R}^2$  with Lipschitz boundary,  $0 < A < |\Omega|$  and  $\alpha < \bar{\alpha}$ . Let  $(u, D)$  be a minimizing configuration. Then the set  $\{u > c\}$  consists of a finite number of connected components whose closures are disjoint. The boundary of each of these connected components consists of finitely many disjoint closed and simple real-analytic curves on which  $|Du| > 0$ . Moreover,  $u$  is analytic in  $\bar{U}$ . We can also construct a set  $\tilde{D}$  such that  $\partial \tilde{D} = \partial U$  and  $\tilde{D}$ ,  $D$  differ only in a zero measure set.*

*Proof.* Assume that there is an infinite number of connected components of  $U$ . Choose a sequence of distinct connected components  $U_i$  of  $U$  and let  $P_i$  be a maximum point of  $u$  in  $U_i$ . Let  $P$  be an accumulating point of  $\{P_i\}$ . Because  $Du(P_i) = 0$  and  $u \in C^{1,1}(\Omega)$ , we have  $Du(P) = 0$ . It is trivial that  $u(P) \geq c$ . Now if  $u(P) > c$ , it means that  $P$  belongs to some connected components of  $U$ , contradicting the fact that each  $P_i$  belongs to a different connected component. So  $P \in \mathcal{F}$  and  $Du(P) = 0$ , contradicting our last lemma.

Let  $P$  be any point on  $\partial U$ . Because  $|Du(P)| > 0$ , there exists some  $r > 0$  such that the set  $B_r(P) \cap \{u > c\}$  is connected. Hence,  $P$  is the boundary point of one and only one connected component of  $U$ . In other words, the closures of any two connected components do not intersect.

Assume  $U_1$  is a connected component of  $U$  such that  $\partial U_1$  consists of infinitely many connected components. Choose a sequence  $\{P_k\}$  such that each  $P_k$  belongs to a connected components  $\mathcal{F}_k$  of  $\partial U_1$  and all  $\mathcal{F}_k$  are distinct. Let  $P \in \partial U_1$  be a limit point of  $\{P_k\}$ . Because  $|Du(P)| > 0$ , there exists  $r > 0$  such that  $B_r(P) \cap \mathcal{F}$  is a simple analytic curve and so, it must belongs to some connected component of  $\partial U_1$ , contradicting the fact that each  $P_k$  belongs to a different component. Thus the boundary of each connected component of  $U$  consists of only a finite number of connected components.

The fact that  $u$  is analytic in  $\bar{U}$  is clear since  $\partial U$  is real-analytic,  $u = c$  on  $\partial U$  and in  $U$ ,  $u$  satisfies the equation

$$-\Delta u = \lambda u.$$

For the existence of  $\tilde{D}$ , just define  $\tilde{D} = {}^c\bar{U}$  and note that  $|\{u = c\}| = 0$ . □

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